Exercise 33

Let L be a nonzero real number.

(a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

Solution

Consider the case where $\lambda = 0$.

$$y'' = 0$$

Solve for y by integrating both sides with respect to x twice.

$$y' = C_1$$

$$y(x) = C_1 x + C_2$$

Apply the boundary conditions in order to determine C_1 and C_2.

$$y(0) = C_2 = 0$$

$$y(L) = C_1 L + C_2 = 0$$

Solving this system yields $C_1 = 0$ and $C_2 = 0$. Therefore, for $\lambda = 0$,

$$\boxed{y(x) = 0.}$$

Consider the case where $\lambda < 0$, that is, $\lambda = -\gamma^2$.

$$y'' - \gamma^2 y = 0$$

This is a homogeneous linear ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} - \gamma^2 (e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - \gamma^2 = 0$$

Solve for r.

$$r = \{-\gamma, \gamma\}$$

Two solutions to the ODE are $e^{-\gamma x}$ and $e^{\gamma x}$. By the principle of superposition, then,

$$y(x) = C_3 e^{-\gamma x} + C_4 e^{\gamma x}.$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 + C_4 = 0$$

$$y(L) = C_3 e^{-\gamma L} + C_4 e^{\gamma L} = 0$$

Solving this system yields $C_3 = 0$ and $C_4 = 0$. Therefore, for $\lambda < 0$,

$$\boxed{y(x) = 0.}$$

Consider the case where $\lambda > 0$, that is, $\lambda = \mu^2$.

$$y'' + \mu^2 y = 0$$

This is a homogeneous linear ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + \mu^2 (e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + \mu^2 = 0$$

Solve for r.

$$r = \{-i\mu, i\mu\}$$

Two solutions to the ODE are $e^{-i\mu x}$ and $e^{i\mu x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_5 e^{-i\mu x} + C_6 e^{i\mu x} \\ &= C_5(\cos \mu x - i \sin \mu x) + C_6(\cos \mu x + i \sin \mu x) \\ &= (C_5 + C_6) \cos \mu x + (-iC_5 + iC_6) \sin \mu x \\ &= C_7 \cos \mu x + C_8 \sin \mu x. \end{aligned}$$

Apply the boundary conditions to determine C_7 and C_8.

$$y(0) = C_7 = 0$$

$$y(L) = C_7 \cos \mu L + C_8 \sin \mu L = 0$$

Since $C_7 = 0$, this second equation reduces to

$$\begin{aligned} C_8 \sin \mu L &= 0 \\ \sin \mu L &= 0 \\ \mu L &= n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \\ \mu &= \frac{n\pi}{L}. \end{aligned}$$

Therefore, for $\lambda = \mu^2 = n^2\pi^2/L^2 > 0$,

$$\boxed{y(x) = C_8 \sin \frac{n\pi x}{L},}$$

where n is an integer and C_8 is arbitrary.